Section 1: Functions, graphs and transformations

Notes and Examples

These notes contain subsections on:

The language of functions

A **mapping** is any rule which associates two sets of items. In a mapping, the **input**, or **object** is something which is to be mapped to something else (the **output**, or **image**).

The set of all possible objects (inputs) of a mapping is called the **domain** of the mapping. The set of outputs for a particular set of inputs for a mapping is called the **range**.

There are four different types of mapping:

- A **one-to-one mapping** is a mapping in which each object is mapped to exactly one image, and each image is the image of exactly one object.

- A **one-to-many mapping** is a mapping in which an object may be mapped to two or more different images.

- A **many-to-one mapping** is a mapping in which two or more particular objects may be mapped to the same image.

- A **many-to-many mapping** is a mapping in which an object may be mapped to two or more different images, and in which two or more objects may be mapped to the same image.

A **function** is a mapping in which there is only one possible image for each object. A function may be one-to-one or many-to-one.

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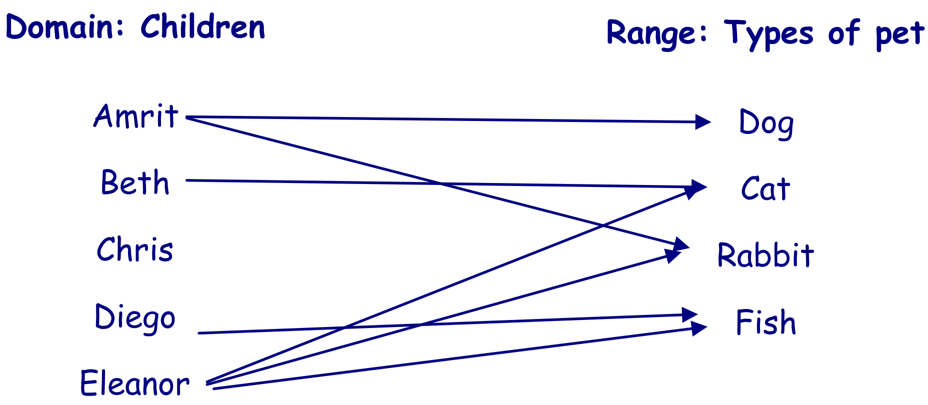
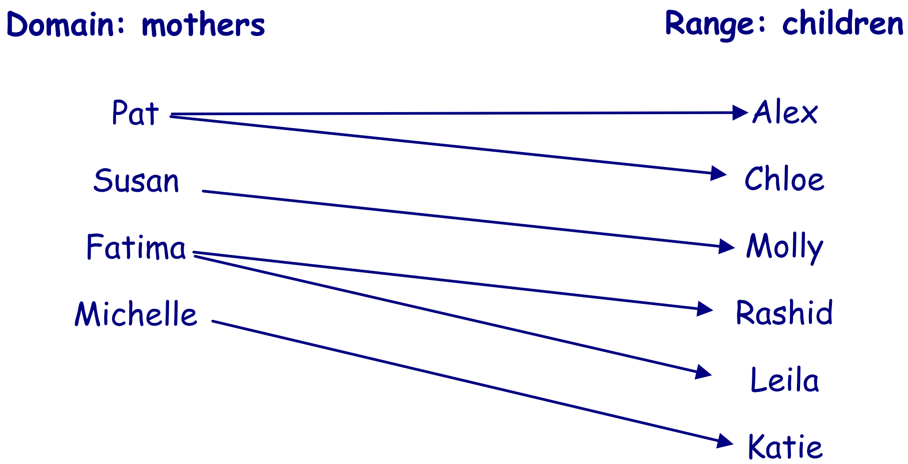
Here are some practical examples of the different types of mappings:

1. Mapping from the set of children who live in a particular street to the set of types of pet.



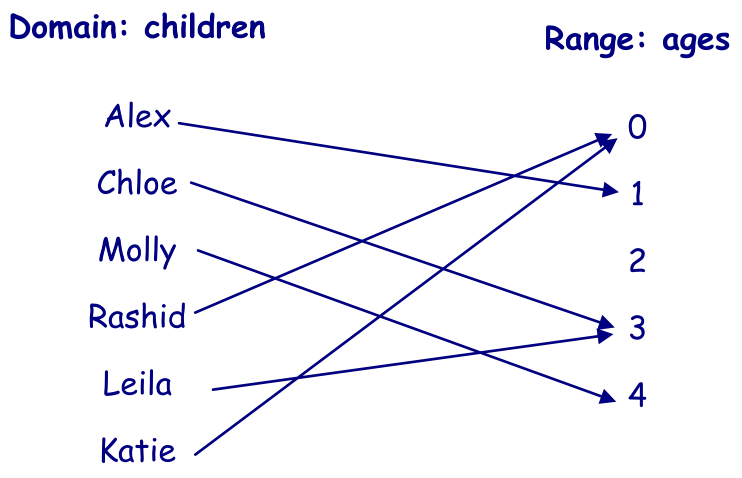
This mapping is many-to-many since a particular child may have more than one pet, and also a particular type of pet can be owned by more than one child.
As only four different types of pet are owned by this particular set of children, the range of the mapping is the set {dog, cat, rabbit, fish}.

2. Mapping from the set of mothers at a toddler group to the set of children at the group.



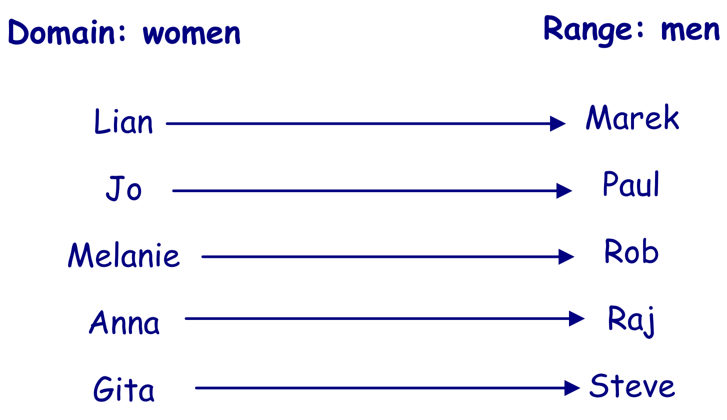
This is a one-to-many mapping: a particular mother may have more than one child, but a particular child has only one mother.
The range is all the children present at the group.

3. Mapping from the set of children at a toddler group to their ages.



This is a many-to-one mapping: a particular child has only one age, but there may be more than one child of a particular age.
Although the set of possible ages for children attending a toddler group might be {0, 1, 2, 3, 4}, there are no 2-year-olds in this particular group, so the range in this case is the set {0, 1, 3, 4}.

4. Mapping from the set of women at a ballroom dancing class to the set of their partners.



This is a one-to-one mapping: each woman has exactly one partner, and each man is the partner of exactly one woman.

The examples above can be helpful to familiarise yourself with the terminology associated with mappings, but in situations like these it is sometimes difficult to define the sets involved clearly and unambiguously (for example, what would you include in types of pet?). However, in mathematical mappings, sets can be defined precisely. Examples of possible domains and ranges include the set of integers, the set of real numbers, or a restricted set of values such as the set of real numbers x for which $0 < x < 1$.

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You may already be familiar with the mathematical “shorthand” that is often used to define particular sets of numbers. Here is a reminder of some of the notation which you may find useful:

$\in$	means “is an element of”
:	means “such that”
$\mathbb{Z}$	denotes the set of integers
$\mathbb{Q}$	denotes the set of rational numbers
$\mathbb{R}$	denotes the set of real numbers
$\mathbb{R}^+$	denotes the set of positive real numbers (similarly for rational numbers, integers etc.)

So, for example,

$x \in \mathbb{R}$	means that x is a real number
$y \in \mathbb{Z} : 0 < y < 10$	means that y is an integer such that y lies between 0 and 10.

Here are some examples of mathematical mappings:

1. $x \to 1-3x, x \in \mathbb{Q}$
   This is a one-to-one mapping. For every value of x, there is one value of $1-3x$, and no two objects map to the same image.
   The range of this mapping is also $\mathbb{Q}$.
   This mapping is also a function as there is only one possible image for each object.

2. $x \to x^2+3, x \in \mathbb{R}$
   This is a many-to-one mapping, since each point has only one possible image, but, for example, -1 and 1 both have image 4.
   All image points are in $\mathbb{R}$, but the image points are all greater or equal to 3, so the range can be written as $y \in \mathbb{R} : y \geq 3$.
   This mapping is also a function as there is only one possible image for each object.

3. $x \to \pm\sqrt{x}, x \in \mathbb{Z}^+$
   This is a one-to-many mapping, since each object has two possible images, but no two objects map to the same image.
   The image points of this mapping are not all integers, but they are all real numbers. The range is not the whole of $\mathbb{R}$, since there are plenty of real numbers which are not the square root of an integer. The range can be written as $y \in \mathbb{R} : y^2 \in \mathbb{Z}$.
   This mapping is not a function as there is more one possible image for each object.

4. $x \to \pm\sqrt{x^2+1}, x \in \mathbb{R}$

This is a many-to-many mapping, since each object has two possible images (e.g. 1 maps to $\sqrt{2}$ and $-\sqrt{2}$) and two objects map to each image (e.g. 1 and -1 both map to $\sqrt{2}$).
The range of this mapping is $\mathbb{R}$.
This mapping is not a function as there is more one possible image for each object.

Using transformations to sketch the graphs of functions

In AS Mathematics you looked at the effect of a translation on the equation of a graph.

You found that:

- $y = \mathrm{f}(x - a)$ is the equation of the graph obtained when the graph of $y = \mathrm{f}(x)$ is moved a units in the positive x direction
- $y = \mathrm{f}(x) + b$ is the equation of the graph obtained when the graph of $y = \mathrm{f}(x)$ is moved b units in the positive y direction
- $y = a\mathrm{f}(x)$ is the equation of the graph obtained when the graph of $y = \mathrm{f}(x)$ is stretched with scale factor a parallel to the y axis
- $y = \mathrm{f}(ax)$ is the equation of the graph obtained when the graph of $y = \mathrm{f}(x)$ is stretched with scale factor $\frac{1}{a}$ parallel to the x axis

You may also have looked at reflections:

- $y = -\mathrm{f}(x)$ is the equation of the graph obtained when the graph of $y = \mathrm{f}(x)$ is reflected in the x axis
- $y = \mathrm{f}(-x)$ is the equation of the graph obtained when the graph of $y = \mathrm{f}(x)$ is reflected in the y axis

In fact reflections are just special cases of the one-way stretches you already know about. The equation $y = -\mathrm{f}(x)$ represents a stretch with scale factor –1 parallel to the y axis, which is the same as reflection in the x axis. The equation $y = \mathrm{f}(-x)$ represents a stretch with scale factor – 1 parallel to the x axis, which is the same as reflection in the y axis.

Summary of transformations of the graph of *y* = f(*x*)

Function	Transformation
$\mathrm{f}(x - a) + b$	Translation through $\begin{pmatrix} a \\ b \end{pmatrix}$
$a\mathrm{f}(x)$	Stretch parallel to y axis, scale factor a
$\mathrm{f}(ax)$	Stretch parallel to x axis, scale factor $\frac{1}{a}$
$-\mathrm{f}(x)$	Reflection in the x axis
$\mathrm{f}(-x)$	Reflection in the y axis

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Successive transformations

In this section you will look at more complicated examples, in which you combine two or more transformations. The main challenge is to make sure that you get the transformations in the correct order: results are sometimes different when you apply transformations in a different order.

Example 1
Explain how the graph $y = 3\sin(2x - 1) + 2$ can be obtained from the graph $y = \sin x$ using successive transformations.

Solution

Since $2x - 1$ means "multiply by 2 then subtract 1, it is very tempting to think that the stretch parallel to the x axis, which results in the $2x$, must come before the translation in the x direction, which results in the - 1. But this order would result in first $y = \sin(2x)$, then $y = \sin(2(x - 1))$.

The order of transformations is as follows:

1.	Transform $y = \sin x$ to $y = 3\sin x$	Stretch scale factor 3 parallel to the y axis
2.	Transform $y = 3\sin x$ to $y = 3\sin(x - 1) + 2$	Translation of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
3.	Transform $y = 3\sin(x - 1) + 2$ to $y = 3\sin(2x - 1) + 2$	Stretch scale factor ½ parallel to the x axis

It is important to be very careful about the order of transformations. In this example, you can do the stretch parallel to the x axis (replacing x by $2x$) at any stage. However, it is essential to do the stretch parallel to the y axis BEFORE the vertical translation – if you do the translation first, you will get $y = 3(\sin 2x + 2)$.

Example 2
The graph $y = x^2$ undergoes the following transformations:

- One way stretch, scale factor 2, parallel to the x axis
- Translation through $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
- One way stretch, scale factor 3, parallel to the y axis

Find the equation of the new graph.

Solution

First apply a one way stretch, scale factor 2, parallel to the *x* axis, to the graph $y = x^2$.

This results in the graph $y=\left(\frac{x}{2}\right)^2$.

Next, apply a translation through $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ to the graph $y=\left(\frac{x}{2}\right)^2$. This results in the graph $y=\left(\frac{x-2}{2}\right)^2-1$.

Finally, apply a one way stretch, scale factor 3, parallel to the *y* axis. This results in the graph $y=3\left(\left(\frac{x-2}{2}\right)^2-1\right)$.

This can be rewritten as:

$$\begin{aligned} y &= 3\left(\frac{x-2}{2}\right)^2-3 \\ &= 3\left(\frac{x^2-4x+4}{4}\right)-3 \\ &= \tfrac{3}{4}x^2-3x+3-3 \\ &= \tfrac{3}{4}x^2-3x \end{aligned}$$