Section 2: Composite and inverse functions

Notes and Examples

These notes contain subsections on

- Composite functions
- Inverse functions

Composite functions

The important thing to remember when finding a composite function is the order in which the functions are written: $\mathrm{fg}(x)$ means first apply the function g to x, then apply the function f to the result.

Example 1

The functions f, g and h are defined by:

$$\mathrm{f}(x) = x + 1$$

$$\mathrm{g}(x) = x^2$$

$$\mathrm{h}(x) = 3x$$

Find the following composite functions:

(i) $\mathrm{fg}(x)$ (ii) $\mathrm{gh}(x)$ (iii) $\mathrm{hgf}(x)$ (iv) $\mathrm{f}^2(x)$

Solution

(i) $\mathrm{fg}(x) = \mathrm{f}[\mathrm{g}(x)]$
$= \mathrm{f}(x^2)$
$= x^2 + 1$

Apply g followed by f;
i.e. square, then add 1.

(ii) $\mathrm{gh}(x) = \mathrm{g}[\mathrm{h}(x)]$
$= \mathrm{g}(3x)$
$= (3x)^2$
$= 9x^2$

Apply h followed by g;
i.e. multiply by 3, then square.

(iii) $\mathrm{hgf}(x) = \mathrm{hg}[\mathrm{f}(x)]$
$= \mathrm{h}[\mathrm{g}(x+1)]$
$= \mathrm{h}[(x+1)^2]$
$= 3(x+1)^2$

Apply f followed by g followed by h;
i.e. add 1, then square, then multiply by 3.

(iv) $\mathrm{f}^2(x) = \mathrm{f}[\mathrm{f}(x)]$
$= \mathrm{f}(x+1)$
$= (x+1) + 1$
$= x + 2$

Apply f twice;
i.e. add 1, then add 1.

Example 2

The functions f and g are defined as:

$$f(x) = \frac{1}{x}$$

$$g(x) = 2x$$

Write the functions:

(i) $\frac{1}{2x}$ (ii) $\frac{2}{x}$ (iii) $4x$

in terms of the functions f and g.

Solution

(i) This function is obtained by first multiplying by 2, then taking the reciprocal; i.e. applying g followed by f. So this function is fg.

(ii) This function is obtained by first taking the reciprocal, then multiplying by 2; i.e. applying f followed by g. So this function is gf.

(iii) This function is obtained by multiplying by 2 twice; i.e. applying g twice. So this function is g^2.

Inverse functions

The inverse of a function 'undoes' the effect of the function. The graph of the inverse function is a reflection of the graph of the original function in the line $y = x$.

Example 3

The function f is defined by $f(x) = 2x^3 + 1$. Find the inverse function f^{-1}.

Solution

The function can be written as: $y = 2x^3 + 1$

Interchanging x and y: $x = 2y^3 + 1$

Rearranging: $x - 1 = 2y^3$

$$\frac{x-1}{2} = y^3$$

$$y = \sqrt[3]{\frac{x-1}{2}}$$

The inverse function is $f^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$.

Remember that to for a function to have an inverse function, it must be one-to-one (you can find the inverse of a many-to-one mapping, but this would be a one-to-many mapping, which is not a function).

Example 4

The functions f and g are defined as:

$$\mathrm{f}(x) = x^2 - 4 \quad x \geq 0$$

$$\mathrm{g}(x) = \sqrt{x-3} \quad x \geq 3$$

(i) What is the range of each function?
(ii) Find the inverse function f^{-1}, stating its domain.
(iii) Find the inverse function g^{-1}, stating its domain.
(iv) Write down the range of f^{-1} and the range of g^{-1}.

Solution

(i) The range of f is $\mathrm{f}(x) \geq -4$.
The range of g is $\mathrm{g}(x) \geq 0$.

(ii) The function can be written as $y = x^2 - 4$
Interchanging x and y: $x = y^2 - 4$
Rearranging: $x + 4 = y^2$
$y = \sqrt{x+4}$
The domain of f^{-1} is the same as the range of f.
$\mathrm{f}^{-1}(x) = \sqrt{x+4} \qquad x \geq -4$

(iii) The function can be written as $y = \sqrt{x-3}$
Interchanging x and y: $x = \sqrt{y-3}$
Rearranging: $x^2 = y - 3$
$y = x^2 + 3$
The domain of g^{-1} is the same as the range of g.
$\mathrm{g}^{-1}(x) = x^2 + 3 \qquad x \geq 0$

(iv) The range of f^{-1} is $\mathrm{f}^{-1}(x) \geq 0$ (the same as the domain of f)
The range of g^{-1} is $\mathrm{g}^{-1}(x) \geq 3$ (the same as the domain of g)