Section 3: The modulus function

Notes and Examples

These notes contain subsections on

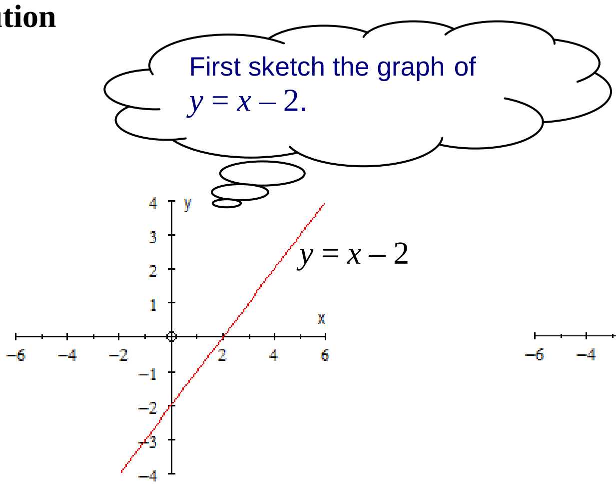
- The modulus function
- Solving equations
- Inequalities

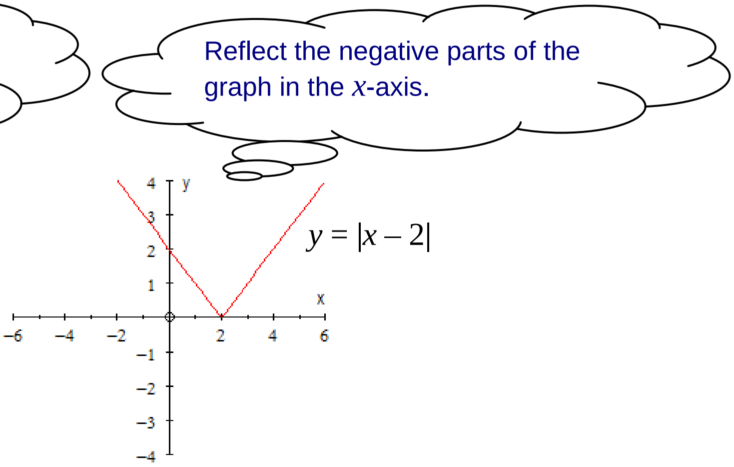
The modulus function

When sketching the graph of $y = |f(x)|$, start by sketching the graph of $y = f(x)$, and then reflect any negative parts of the graph in the x-axis.

Example 1
Sketch the graphs of the functions:
(i) $y = |x - 2|$
(ii) $y = |2x + 1| - 1$

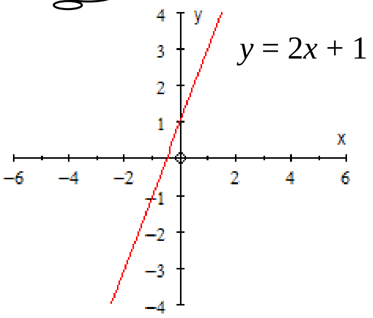
Solution
(i)



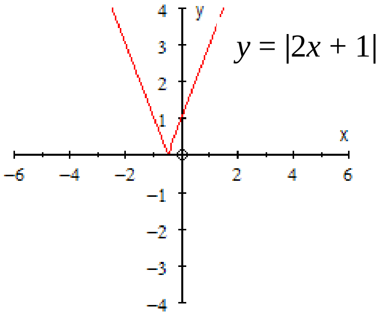


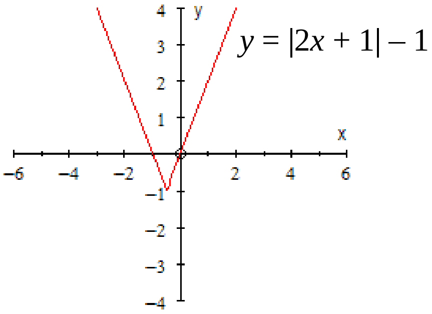
(ii)

First sketch the graph of $y = 2x + 1$.



Reflect the negative parts of the graph in the x-axis.





Finally translate the graph vertically downwards by 1 unit.

Any inequality of the form $p < x < q$ can be written in the form $|x - a| < b$. This is basically reversing the process of solving an inequality.

Example 2

Write the inequality $6 \le x \le 14$ in the form $|x - a| \le b$.

Solution

$$|x-a| \le b \Rightarrow -b \le x-a \le b$$

$$\Rightarrow -b+a \le x \le b+a$$

$$-b+a = 6$$

$$b+a = 14$$

Adding: $2a = 20$

$$a = 10$$

$$b = 4$$

The inequality can be written as $|x - 10| \le 4$.

Solving equations

Graphs are often useful if you need to solve an equation which involves a modulus function.

Example 3

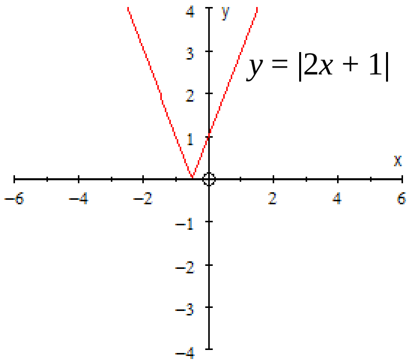
(i) Sketch the graph of $y = |2x + 1|$.

(ii) Hence, or otherwise, solve the equation $|2x + 1| = x + 2$.

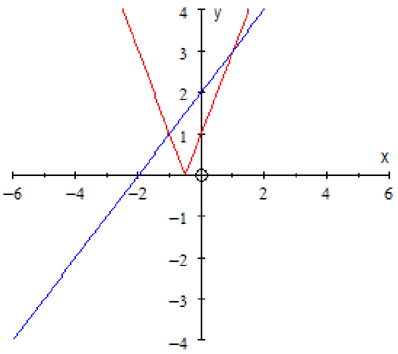
(iii) Solve the equation $|2x + 1| = 2 - 3x$.

Solution

(i)



(ii) By sketching the graphs of $y = |2x + 1|$ and $y = x + 2$ on the same axes, you can see that there are two roots.



$$2x + 1 = x + 2$$
$$x = 1$$

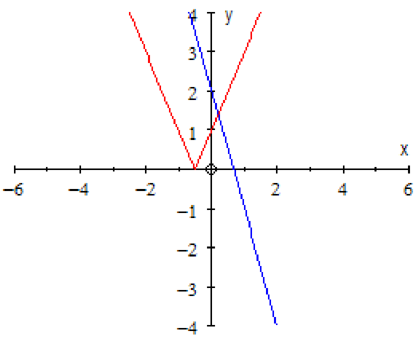
One of the roots involves the part of $y = |2x + 1|$ where $2x + 1$ is positive, i.e. where the graph is $y = 2x + 1$

$$-(2x + 1) = x + 2$$
$$-2x - 1 = x + 2$$
$$3x = -3$$
$$x = -1$$

The other root involves the part of $y = |2x + 1|$ where $2x + 1$ is negative, i.e. where the graph is $y = -(2x + 1)$

The roots are $x = 1$ and $x = -1$.

(iii) By sketching the graphs of $y = |2x + 1|$ and $y = 2 - 3x$ on the same axes, you can see that there is only one root, which involves the part of $y = |2x + 1|$ where $2x + 1$ is positive. The line $y = 2 - 3x$ will not cross the part of $y = |2x + 1|$ where $2x + 1$ is negative, since the line $y = 2 - 3x$ has the steeper gradient.



$2x+1=2-3x$

$5x=1$

$x=\frac{1}{5}$

The root is $x=\frac{1}{5}$.

Equations like this can also be solved algebraically by squaring both sides, however care must be taken because false roots can be found.

For example, using the equation from Example 3(iii) above:

$$|2x+1|=2-3x$$
$$(2x+1)^2=(2-3x)^2$$
$$4x^2+4x+1=4-12x+9x^2$$
$$5x^2-16x+3=0$$
$$(x-3)(5x-1)=0$$
$$x=3 \text{ or } \tfrac{1}{5}$$

Putting $x=\frac{1}{5}$ into the equation gives LHS $=\frac{7}{5}$, RHS $=\frac{7}{5}$
However, putting $x=3$ into the equation gives LHS $=7$, RHS $=-7$

By looking at the graph you can see that $x=3$ is not a root, but is the point where the left-hand side of the graph of $y=|2x+1|$ would meet the line $y=2-3x$ if it were extended.

Using a graph helps to avoid this kind of mistake. If using an algebraic approach, always check by putting the roots into the equation.

Inequalities

To solve inequalities which involve a modulus sign, you need to consider whether the solution set involves two separate intervals or just one.

An inequality of the form $|f(x)| < a$ can be written as $-a < f(x) < a$.
An inequality of the form $|f(x)| > a$ can be written as $f(x) < -a$ or $f(x) > a$.

The inequality can then be solved using the usual rules for inequalities.

Example 4
Solve the following inequalities.

(i) $|2x+1|\le 7$

(ii) $2|x-2|+1>3$

(iii) $|1-3x|<4$

Solution

(i) $|2x+1| \le 7$

$-7 \le 2x+1 \le 7$

$-8 \le 2x \le 6$

$-4 \le x \le 3$

(ii) $2|x-2|+1 > 3$

$2|x-2| > 2$

$|x-2| > 1$

$x-2 < -1$ or $x-2 > 1$

$x < 1$ or $x > 3$

(iii) $|1-3x| < 4$

$|3x-1| < 4$

$|1 - 3x|$ is equal to $|3x - 1|$. It is easier to avoid having a negative term in x.

$-4 < 3x-1 < 4$

$-3 < 3x < 5$

$-1 < x < \frac{5}{3}$

Inequalities like the ones above are quite straightforward to solve algebraically. However, for more complicated inequalities, you should use a graph to solve the associated equation, and then deduce the solution to the inequality from the graph.

The next example is based on Example 3, but involves inequalities rather than equations.

Example 5

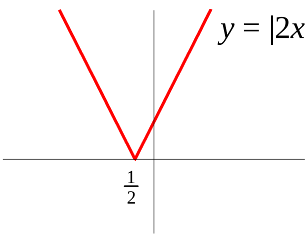
(i) Sketch the graph of $y = |2x + 1|$.

(ii) Hence, or otherwise, solve the inequality $|2x + 1| < x + 2$.

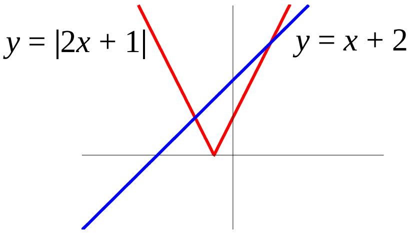
(iii) Solve the equation $|2x + 1| > 2 - 3x$.

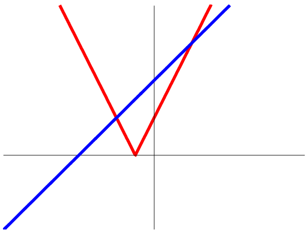
Solution

(i)



(ii) By sketching the graphs of $y = |2x + 1|$ and $y = x + 2$ on the same axes, you can see that there are two intersection points.



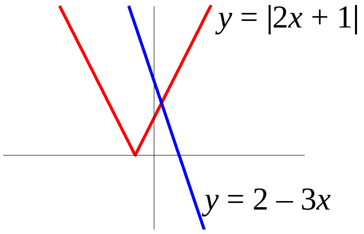


$$2x+1=x+2$$
$$x=1$$

$$-(2x+1)=x+2$$
$$-2x-1=x+2$$
$$3x=-3$$
$$x=-1$$

The solution to the inequality $|2x + 1| < x + 2$ is the range of values for x for which the red graph lies below the blue graph.
The solution is $-1 < x < 1$.

(iii)



$$2x+1=2-3x$$
$$5x=1$$
$$x=\tfrac{1}{5}$$

The solution to the inequality $|2x + 1| > 2 - 3x$ is the range of values for x for which the red graph lies above the blue graph.
The solution is $x > \tfrac{1}{5}$.