Section 3: Partial fractions

Notes and Examples

These notes contain subsections on

- Fractions with linear factors in the denominator
- Fractions with repeated linear factors in the denominator
- Using partial fractions with the binomial expansion

In the previous section, you practised adding and subtracting algebraic fractions.

e.g. $\frac{1}{x+1}+\frac{3}{x-2}=\frac{4x+1}{(x+1)(x-2)}$.

Sometimes it is useful to reverse this process: i.e. to express a complicated fraction as the sum of two or more simpler ones. This process is called finding partial fractions.

You will be looking only at proper fractions: i.e. algebraic fractions where the degree (or order) of the numerator is less than the degree of the denominator.

Remember that the degree of a polynomial expression is the highest power of x in the expression.

Fractions with linear factors in the denominator

Any fraction of the form $\frac{px+q}{(ax+b)(cx+d)}$ can be written in the partial fractions form $\frac{A}{ax+b}+\frac{B}{cx+d}$.

There cannot be any terms in x in the numerators of the partial fractions, as this would mean that the fractions would be improper.

To find the unknown constants A and B in the partial fractions, you first multiply through by the common denominator to clear the fractions.
There are then two basic methods to find the constants:

- substitute any two values for x to give two equations in A and B (by choosing the values carefully you can make this very easy)
- multiply out and equate coefficients.

Substitution is often the most efficient method, but in some cases equating coefficients may be an easier way to find one or more of the unknown constants.

Example 1

Express $\dfrac{x-9}{(x+1)(2x-3)}$ in partial fractions.

Solution

The partial fractions must be of the form $\dfrac{A}{x+1}+\dfrac{B}{2x-3}$.

So $\dfrac{x-9}{(x+1)(2x-3)}=\dfrac{A}{x+1}+\dfrac{B}{2x-3}$.

Multiplying both sides by $(x + 1)(2x - 3)$: $x - 9 = A(2x - 3) + B(x + 1)$

This equation is true for **all** possible values of x.
This means that you can substitute any number for x and get a true statement. Although any number will do, you can make it easier by choosing carefully. By choosing $x = -1$, the term in B will disappear, making it easy to find A.

Substitute $x = -1$ $-10 = -5A$

$A = 2$

To find B, you could choose $x = \frac{3}{2}$ so that the term in A disappears. However, if you prefer to avoid using fractions, you can choose any value for x and use the fact that $A = 2$. Perhaps the simplest choice is $x = 0$.

Substitute $x = 0$ $-9 = 2(-3) + B$

$-9 = -6 + B$

$B = -3$

Hence $\dfrac{x-9}{(x+1)(2x-3)}=\dfrac{2}{x+1}-\dfrac{3}{2x-3}$

In some cases you may need to factorise the denominator before you start.

Example 2

Express $\dfrac{1}{x^2+x-2}$ in partial fractions.

Solution

First factorise the denominator: $\dfrac{1}{x^2+x-2}=\dfrac{1}{(x-1)(x+2)}$.

$$\frac{1}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$$

Multiplying through by $(x-1)(x+2)$: $1 = A(x+2) + B(x-1)$

Substitute $x = 1$ $\quad 1 = 3A$

$A=\frac{1}{3}$

Substitute $x = -2$ $\quad 1 = -3B$

$B=-\frac{1}{3}$

Hence $\dfrac{1}{(x-1)(x+2)}=\dfrac{1}{3(x-1)}-\dfrac{1}{3(x+2)}$

Fractions with repeated linear factors in the denominator

This method needs to be adapted slightly when there is a repeated linear factor in the denominator. Trying to write $\dfrac{px+q}{(ax+b)^2}$ in the form $\dfrac{A}{ax+b}+\dfrac{B}{ax+b}$ does not work as this will just give $\dfrac{px+q}{(ax+b)^2}=\dfrac{A+B}{ax+b}$.

However, any fraction of the form $\dfrac{px+q}{(ax+b)^2}$ can be written in the form $\dfrac{A}{ax+b}+\dfrac{B}{(ax+b)^2}$.

Example 3

Express $\dfrac{3x-1}{x^2-4x+4}$ in partial fractions.

Solution

Factorising the denominator: $\dfrac{3x-1}{x^2-4x+4}=\dfrac{3x-1}{(x-2)^2}$

$$\frac{3x-1}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$$

Multiplying through by $(x-2)^2$: $3x-1 = A(x-2) + B$

Substituting $x = 2$: $\quad 5 = B$

Equating coefficients of x: $\quad 3 = A$

Hence $\dfrac{3x-1}{(x-2)^2}=\dfrac{3}{x-2}+\dfrac{5}{(x-2)^2}$

If there are other linear factors in the denominator, then treat them in the same way as before. In general, any fraction of the form $\frac{px^2+qx+r}{(ax+b)^2(cx+d)}$ can be written in the form $\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{cx+d}$, and this can be extended to any number of factors in the denominator.

Example 4

Express $\frac{2x^2+7}{(2x+1)^2(x+3)}$ in partial fractions.

Solution

$$\frac{2x^2+7}{(2x+1)^2(x+3)}=\frac{A}{2x+1}+\frac{B}{(2x+1)^2}+\frac{C}{x+3}$$

Multiplying through by $(2x + 1)^2(x + 3)$:

$$2x^2 + 7 = A(2x + 1)(x + 3) + B(x + 3) + C(2x + 1)^2$$

Substituting $x = -3$: $18 + 7 = C(-6 + 1)^2$

$25 = 25C$

$C = 1$

Substituting $x=-\frac{1}{2}$: $2\times\frac{1}{4}+7=B(-\frac{1}{2}+3)$

$\frac{15}{2}=\frac{5}{2}B$

$B = 3$

Substituting $x = 0$: $7 = 3A + 3B + C$

$7 = 3A + 9 + 1$

$3A = -3$

$A = -1$

Hence $\frac{2x^2+7}{(2x+1)^2(x+3)}=-\frac{1}{2x+1}+\frac{3}{(2x+1)^2}+\frac{1}{x+3}$

Using partial fractions with the binomial expansion

The following example shows how partial fractions can be used to simplify the working if you want to find a binomial expansion.

For example, if you want to find the binomial expansion of $\frac{1}{(x-1)(2-x)}$ up to the term in x^3, you could write it as $(x-1)^{-1}(2-x)^{-1}$, apply the binomial expansion to each bracket separately and then multiply out the brackets, discarding any terms of order greater than x^3. However, Example 6 shows how using partial fractions can make the work much easier, as in this case you have to add two algebraic expressions rather than multiplying.

Example 5

Expand $\dfrac{1}{(x-1)(2-x)}$ up to the term in x^3, stating the range of values of x for which the expansion is valid.

Solution

$$\frac{1}{(x-1)(2-x)} = \frac{A}{x-1} + \frac{B}{2-x}$$

Multiplying through by $(x-1)(2-x)$: $1 = A(2-x) + B(x-1)$

Substituting $x = 2$: $1 = B$

Substituting $x = 1$: $1 = A$

$$\begin{aligned}\frac{1}{(x-1)(2-x)} &= \frac{1}{x-1} + \frac{1}{2-x} \\ &= (x-1)^{-1} + (2-x)^{-1} \\ &= -(1-x)^{-1} + 2^{-1}\left(1-\tfrac{x}{2}\right)^{-1}\end{aligned}$$

> Both parts must be written in the form $(1 \pm kx)^{-1}$ before the expansion can be carried out.

$$\begin{aligned}(1-x)^{-1} &= 1 + (-1)(-x) + \frac{(-1)(-2)}{1\times 2}(-x)^2 + \frac{(-1)(-2)(-3)}{1\times 2\times 3}(-x)^3 + \ldots \\ &= 1 + x + x^2 + x^3 + \ldots\end{aligned}$$

> Expand the two parts separately

This expansion is valid for $-1 < x < 1$.

$$\begin{aligned}\left(1-\tfrac{x}{2}\right)^{-1} &= 1 + (-1)\left(-\tfrac{x}{2}\right) + \frac{(-1)(-2)}{1\times 2}\left(-\tfrac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{1\times 2\times 3}\left(-\tfrac{x}{2}\right)^3 + \ldots \\ &= 1 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 + \tfrac{1}{8}x^3 + \ldots\end{aligned}$$

This expansion is valid for $-1 < \frac{x}{2} < 1 \Rightarrow -2 < x < 2$

$$\begin{aligned}\frac{1}{(x-1)(2-x)} &= -(1-x)^{-1} + \tfrac{1}{2}\left(1-\tfrac{x}{2}\right)^{-1} \\ &= -\left(1 + x + x^2 + x^3 + \ldots\right) + \tfrac{1}{2}\left(1 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 + \tfrac{1}{8}x^3 + \ldots\right) \\ &= -1 - x - x^2 - x^3 + \tfrac{1}{2} + \tfrac{1}{4}x + \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \ldots \\ &= -\tfrac{1}{2} - \tfrac{3}{4}x - \tfrac{7}{8}x^2 - \tfrac{15}{16}x^3 + \ldots\end{aligned}$$

The expansion is valid for $-1 < x < 1$.

> The expansion is valid for values of x for which both parts are valid, i.e. both $-1 < x < 1$ and $-2 < x < 2$ must be true.

Note: another application of partial fractions is in integration. You will look at this in a later topic.