Section 2: Rational expressions

Notes and Examples

These notes contain subsections on

Simplifying algebraic fractions

You are familiar with the idea of "cancelling" to simplify numerical fractions: for example, $\frac{9}{12}$ can be simplified to $\frac{3}{4}$ by dividing both the numerator and the denominator by 3. The same technique can be used in algebra. Remember that "cancelling" involves dividing, not subtracting.

Example 1

Simplify $\dfrac{6xy^3 + 2x^2y}{10x^2y}$

It is very important to remember that you can only "cancel" if you can divide each term in both the numerator and denominator by the same expression. In this case, don't be tempted to divide by $2x^2y$ – although this is a factor of both $2x^2y$ and $10x^2y$, it is not a factor of $6xy^3$. In a case like this, it is best to factorise the top first, so that it is easier to see the factors.

Solution

$$\frac{6xy^3 + 2x^2y}{10x^2y} = \frac{2xy(3y^2 + x)}{10x^2y}$$
$$= \frac{3y^2 + x}{5x}$$

$2xy$ is a common factor of both top and bottom

Multiplying and dividing algebraic fractions

As with numerical fractions, you multiply algebraic fractions by multiplying the numerators and multiplying the denominators.

Sometimes you can cancel before carrying out a multiplication, to make the numbers simpler: e.g. $\frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{9}^{3}}{\cancel{4}_{2}} = \frac{3}{2}$. You can do the same with algebraic fractions.

When multiplying, remember to use brackets where appropriate.

Example 2

Simplify $\dfrac{3a}{a+1} \times \dfrac{2a+2}{a+2}$

Solution

Again, factorise where possible first.

$$\frac{3a}{a+1} \times \frac{2a+2}{a+2} = \frac{3a}{\cancel{a+1}} \times \frac{2(\cancel{a+1})}{a+2}$$

$$= \frac{6a}{a+2}$$

(a + 1) is a common factor of both top and bottom

Notice that you cannot cancel a here, as it is not a factor of a + 2.

Remember that to divide fractions, you take the reciprocal of the second fraction and then multiply, in the same way as you do for numerical fractions:

$$\frac{2}{3} \div \frac{4}{7} = \frac{\cancel{2}}{3} \times \frac{7}{\cancel{4}_2} = \frac{7}{6}$$

Example 3

Simplify $\dfrac{x+1}{x^2} \div \dfrac{x^2-1}{2x}$

Solution

$$\frac{x+1}{x^2} \div \frac{x^2-1}{2x} = \frac{x+1}{x^2} \times \frac{2x}{x^2-1}$$

$$= \frac{\cancel{x+1}}{x^{\cancel{2}}} \times \frac{2\cancel{x}}{(x-1)(\cancel{x+1})}$$

$$= \frac{2}{x(x-1)}$$

(x + 1) and x are both factors of both the top and the bottom

Adding and subtracting algebraic fractions

Algebraic fractions follow the same rules as numerical fractions. When adding or subtracting, you need to find the common denominator, which may be a number or an algebraic expression.

Example 4

Simplify

(i) $\dfrac{2x}{3} + \dfrac{x}{4} - \dfrac{5x}{6}$

(ii) $\dfrac{1}{2x} - \dfrac{1}{x^2}$

Solution

(i) The common denominator is 12, as 3, 4 and 6 are all factors of 12.

$$\frac{2x}{3}+\frac{x}{4}-\frac{5x}{6}=\frac{8x}{12}+\frac{3x}{12}-\frac{10x}{12}$$
$$=\frac{8x+3x-10x}{12}$$
$$=\frac{x}{12}$$

(ii) The common denominator is $2x^2$.

$$\frac{1}{2x}-\frac{1}{x^2}=\frac{x}{2x^2}-\frac{2}{2x^2}$$
$$=\frac{x-2}{2x^2}$$

Algebraic division

You have already met polynomial division in the context of solving polynomial equations such as cubic equations. Here you will look at dividing a polynomial by a linear or quadratic polynomial. There may be a remainder when you divide.

The first important thing that you must grasp about algebraic division is about the order of the polynomials. If you divide a cubic polynomial (degree 3) by a linear expression (degree 1) then the quotient has degree at most 2 and the remainder (if there is one) is just a number (degree 0). If you divide a polynomial of degree 5 by a quadratic (degree 2) then the quotient has at most degree 3 and the remainder (if there is one) is at most linear (degree 1). In other words, if a polynomial of degree n is divided by a polynomial of degree r, then the quotient has degree at most $n-r$ and the remainder has degree at most $r-1$.

There are a number of different techniques which can be used. Three of the common techniques are shown in Example 5.

Example 5

Divide $2x^3-3x^2+x-3$ by $x-2$

Solution 1: Long division

The quotient will go here

$$x-2\,\overline{)\,2x^3-3x^2+x-3}$$

$$
\begin{array}{r}
2x^2 \phantom{{}-3x^2+x-3} \\
x-2\,\big)\overline{2x^3-3x^2+x-3} \\
\underline{2x^3-4x^2} \phantom{{}+x-3} \\
x^2 \phantom{{}+x-3}
\end{array}
$$

Divide the leading term of the dividend by the leading term of the divisor. In this case, $2x^3 \div x = 2x^2$

Multiply the divisor, $x - 2$, by $2x^2$ …

… and subtract

$$
\begin{array}{r}
2x^2+x \phantom{{}-3} \\
x-2\,\big)\overline{2x^3-3x^2+x-3} \\
\underline{2x^3-4x^2} \phantom{{}+x-3} \\
x^2+x \phantom{{}-3} \\
\underline{x^2-2x} \phantom{{}-3} \\
3x \phantom{{}-3}
\end{array}
$$

Divide the leading term of the remainder, $x^2 + x$, by the leading term of the divisor. In this case, $x^2 \div x = x$

Multiply the divisor, $x - 2$, by x …

Bring down the next term, x

… and subtract

$$
\begin{array}{r}
2x^2+x+3 \\
x-2\,\big)\overline{2x^3-3x^2+x-3} \\
\underline{2x^3-4x^2} \phantom{{}+x-3} \\
x^2+x \phantom{{}-3} \\
\underline{x^2-2x} \phantom{{}-3} \\
3x-3 \\
\underline{3x-6} \\
3
\end{array}
$$

Divide the leading term of the remainder, $3x - 3$, by the leading term of the divisor. In this case, $3x \div x = 3$

Multiply the divisor, $x - 2$, by 3 …

Bring down the next term, -3

… and subtract

The quotient is $2x^2 + x + 3$ and the remainder is 3.

Solution 2: Equating coefficients

Since a cubic polynomial is being divided by a linear polynomial, the quotient is quadratic and the remainder is just a number.

$Ax^2 + Bx + C$ is the quotient

D is the remainder

$$\frac{2x^3-3x^2+x-3}{x-2} = Ax^2+Bx+C+\frac{D}{x-2}$$

Multiplying through by x - 2

$$
\begin{aligned}
2x^3-3x^2+x-3 &= \left(Ax^2+Bx+C\right)(x-2)+D \\
&= Ax^3+(B-2A)x^2+(C-2B)x-2C+D
\end{aligned}
$$

Compare the left and right hand sides of this equation

Equating coefficients of x^3: $A = 2$

Equating coefficients of x^2: $B - 2A = -3 \quad \Rightarrow B = 1$

Equating coefficients of x: $C - 2B = 1 \quad \Rightarrow C = 3$

Equating constant terms: $-2C + D = -3 \quad \Rightarrow D = 3$

$$\frac{2x^3 - 3x^2 + x - 3}{x - 2} = 2x^2 + x + 3 + \frac{3}{x - 2}$$

The quotient is $2x^2 + x + 3$ and the remainder is 3.

Solution 3: Inspection

Since a cubic polynomial is being divided by a linear polynomial, the quotient is quadratic and the remainder is just a number.

This will be a quadratic

This will be the remainder

$$2x^3 - 3x^2 + x - 3 = (x - 2)(..............) + ...$$

$2x^2$ must go here to give the $2x^3$ term

$$2x^3 - 3x^2 + x - 3 = (x - 2)(2x^2..............) + ...$$

Think about the x^2 term. You already have $-4x^2$, so you need x^2 to make $-3x^2$

$$2x^3 - 3x^2 + x - 3 = (x - 2)(2x^2 + x.........) + ...$$

Think about the x term. You already have $-2x$, so you need $3x$ to make x

$$2x^3 - 3x^2 + x - 3 = (x - 2)(2x^2 + x + 3) + ...$$

Think about the constant term. You already have -6, so you need 3 to make -3.

$$2x^3 - 3x^2 + x - 3 = (x - 2)(2x^2 + x + 3) + 3$$

In practice you can write this out in just one line.

The quotient is $2x^2 + x + 3$ and the remainder is 3.

You can divide any polynomial by another so long as the order of the dividend is greater than or equal to the order of the divisor. Example 6 shows a situation in which the dividend and divisor are both linear. This examples are shown using the method of equating coefficients: you can try them with the other methods.

Example 6

(i) Divide $2x - 5$ by $x - 3$.

(ii) Hence sketch the graph of $y = \dfrac{2x - 5}{x - 3}$

Solution

(i) Since a linear polynomial is being divided by a linear polynomial, the quotient is a constant and the remainder is a constant.

$$\frac{2x-5}{x-3} = A + \frac{B}{x-3}$$
$$2x-5 = A(x-3) + B$$
$$= Ax - 3A + B$$

Equating coefficients of x: $A = 2$
Equating constant terms: $-3A + B = -5 \quad \Rightarrow B = 1$

$$\frac{2x-5}{x-3} = 2 + \frac{1}{x-3}$$

(ii) $y = \dfrac{2x-5}{x-3} = 2 + \dfrac{1}{x-3}$

This can be sketched by starting from the graph $y = \dfrac{1}{x}$ and translating 3 units to the right and 2 units up.