**1.1 Proof**

You should be able to use three different types of proof.

**Proof by deduction**

* + A proof by deduction (or by reasoning) always has a few logical steps.

**Proof by exhaustion**

* + In a proof by exhaustion you check all possible cases separately.

**Disproof by counter-example**

* + When trying to find a counter-example, you may need to try several possibilities.

You can use the symbols  in proof questions.

 means **implies**.   
 For example, *x* + 2 = 5  *x* = 3.

 means **is equivalent to**.   
 For example,    
 You can read this as   
  and 

 means **is implied by**.   
 This is not often used, but an example is .

Examples

**Example 1 Proof by deduction**  
Use completing the square to prove that ** is positive for all values of *n*.

|  |  |
| --- | --- |
| for all values of *n*.  Therefore is positive for all values of *n*. | First complete the square.  Remember that squares cannot be negative.  With a proof you need a conclusion. |

**Example 2 Proof by exhaustion** Prove that 2*n* < (*n* + 1)2 for all positive integers *n* < 6.

|  |  |
| --- | --- |
| The only positive integers less than 6 are 1, 2, 3, 4 and 5.  Check that 2*n* < (*n* + 1)2 for each case.  *n* = 1:  2*n* = 21 = **2** and (*n* + 1)2 = (1 + 1)2 = **4**; **2** < **4**  *n* = 2:  2*n* = 22 = **4** and (*n* + 1)2 = (2 + 1)2 = **9**; **4** < **9**  *n* = 3:  2*n* = 23 = **8** and (*n* + 1)2 = (3 + 1)2 = **16**; **8** < **16**  *n* = 4:  2*n* = 24 = **16** and (*n* + 1)2 = (4 + 1)2 = **25**; **16** < **25**  *n* = 5:  2*n* = 25 = **32** and (*n* + 1)2 = (5 + 1)2 = **36**; **32** < **36**  2*n* < (*n* + 1)2 for all five possible cases and this completes the proof. | It may be helpful to begin by explaining what you will do to prove the statement.  In a proof by exhaustion, work through all the possibilities in a systematic way to make sure you don’t miss any.  Always end your proof with a conclusion. |

**Example 3** Wemin states the following.

A number of the form *n*2 + 3*n* + 7is a prime number for any integer greater than 0.

Prove that Wemin is wrong by finding a counter-example.

|  |  |
| --- | --- |
| Try *n* = 1  =  This is prime.  Try *n* = 2  =  This is prime.  Try *n* = 3 =  This is *not* prime, as 25 can be divided by 5.  This proves that Wemin is wrong.  *n*= 3 is a counter-example. | You may have to try several values before you find a counter-example.  Always end your proof with a conclusion. |

**Note:** You may have spotted that *n* = 7 is another counter-example.   
When *n* = 7, each of the three terms of *n*2 + 3*n* + 7 is a multiple of 7 and so the sum is also a multiple of 7. (72 + 3 × 7 + 7 = 11 × 7)

**Example 4** Find a counter-example to disprove the statement   
  for integers *m* and *n* (*m* ≠ 0 and *n* ≠ 0).

|  |  |
| --- | --- |
| For *m* = 2 and *n* = –1,  and  So  but | You can pick any *m* > 0 and *n* < 0.  Always finish with a statement. You could add ‘This proves that the statement is not correct.’ |

Exercise

**1.** Prove that  is positive for all values of *x*.

**2.** Prove that  is positive for all values of *x*.

**3. (a)** Prove that has a factor 4 for all even numbers *n*.

**(b)** Prove by counter-example that does not have a factor 4 for all integers *n*.

**4.** Find a counter-example to prove that  is not prime for all positive integer values of *n*.

**5.** Prove that .

**6.** Prove by exhaustion that there are exactly two natural numbers less than 101 that are both a square and a cube integer. [Note: the natural numbers are the set {1, 2, 3, … }.]

**7.** Find a counter-example to disprove the following statement.



**8.** Disprove the following statement.



**9.** Prove that  is not prime for all positive integer values of *n*.

**10.** Insert one of  into the following statement about integer *n*. Explain your answer.

*n*2 + 1 is even ……. *n* is odd

**11.** Insert one of  into the following statement.

*m*2 is an integer ……. *m* is an integer

Answers

**1.**   
 (*x* + 4)2 ≥ 0 for all values of *x*, and therefore (*x* +4)2 + 5 > 0.  
 Hence is positive for all values of *x*.

**2.** ;  so    
 Therefore for all values of *x*.

**3. (a)** *n* is even  where *m* is an integer.



So is a multiple of 4.

**(b)** A counter-example is given by any odd value of *n*.

For example, for *n* = 1, . This is not a multiple of 4.

**4.** *n* = 11 is a counter-example.

You may have found a different counter-example, such as *n* = 10.



Therefore  is not always prime.

With a proof by deduction like this, you start with the expression on one side and keep working until you get to the expression on the other side.

**5.** 

  = 

, which is what you needed to prove.

**6.** The cubes less than 101 are 13 = 1, 23 = 8, 33 = 27 and 43 = 64. (53 = 125 > 101)

Of these four numbers, two are squares, 12 = 1 and 82 = 64.

So there are exactly two cubes that are also squares between 0 and 101.

**7.** When . Therefore  is not correct.

So  is not correct.

**8.** ** gives a counter-example. 

This proves that the statement is false.

**9.** *n* = 13 gives a counter-example.



Therefore is not prime for all values of *n*.

**10.** because

*n*2 + 1 is even  *n*2 is odd  *n* is odd

and *n* is odd  *n*2 is odd  *n*2 + 1 is even

**11.**  because when *m* is an integer *m × m* is an integer.

However *m*2 = 3 gives *m* = , so when *m*2 is an integer, *m* is not always an integer.