**6.7 Exponential growth and decay**

You need to be familiar with

* exponential models
* the term ‘initial’, meaning when *t* = 0
* the behaviour of the model for large values of *t*
* considering limitations or refinements of exponential models.

It will be helpful to be familiar with the following graphs.

 *y* = e*t* and *y* = e–*t*

* In modelling *t* is usually positive.
* The curves approach the horizontal axis,
which is an asymptote for both curves.
* When *t* = 0, *y* = 1



 *y* = *a* + *b*e–*t* and *y* = *a* – *b*e–*t*

* For each curve the line *y* = *a*
is an asymptote, which means
when *t* tends to infinity, *y* tends to *a*.

Examples

**Example 1** The owner of a certain website models the number of visits to the site in a day by the exponential equation *n*=320e*t*, where *n* is the number of visits on day number *t*.

**(a)** According to the model, how many visits were made on the first day?

**(b)** According to the model, how many visits were made on the seventh day?

**(c)** Comment on the long-term suitability of the model.

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| --- | --- |
| **(a)** *t* = 0, so *n* = 320 × e1  = 870 So 870 visits were made.**(b)** *t* = 7, so *n* = 320 × e7 = 350 992 So about 350 000 visits were made.**(c)** The model is not suitable for large values of *t*, as the number of visits grows too fast. (For example the model suggests that on day 20 there will be over 10 billion visits, which is not realistic.)  | *t* = 1 on the first day.Always answer in context.Substitute *t* = 7 for the seventh day.Again, answer in the context of the question. Don’t just give a number. |

**Note:** If you were asked to refine this model, you could simply state that after the seventh day the number of visits in a day remain constant at 350 000.
You write this as *n*=320e*t* for *t* ≤ 7 and *n* = 350 000 for *t* > 7.

**Example 2 Radioactive decay**A radioactive material decays and its mass is modelled by the equation *M = D*e*–kt*, where *M* is the mass in grams and *t* is the number of years since the first measurement. The initial mass of 40.0 grams of the material decays to 12.4 grams in 10 years.

**(a)** Find the positive constants *D* and *k*.

**(b)** After how many years does only 0.1 gram of the material remain?

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| **(a)** *D* = 40 (or 40.0)  12.4 = 40 e*–k*× 10 0.31 = e*–k*× 10ln 0.31 = –*k* × 10 so *k* = –  = 0.117 | *D* is the initial amount (substitute *t* = 0). Substitute 12.4, 40 and 10.Divide both sides by 40.Take logarithms on both sides.Divide by –10. |
| **(b)** Use the model *M =* 40 e*–*0.117*t* and substitute 0.1 for *M*. 0.1 = 40 e*–*0.117*t*0.0025 = e*–*0.117*t*ln 0.0025 = –0.117*t* so *t* = 51.2  After 51.2 years only 0.1 gram remains. | Divide both sides by 40.Take logarithms on both sides and divide by –0.117Give your answer in context. |

**Example 3 Compound and continuous compound interest**

 *A* is the amount in a bank account, *P* is the principal or initial amount invested, *n* is the number of months *P* is invested, and *r*% is the monthly interest rate.

**(a)** Use the formula for compound interest to find the amount in the bank account after 8 months when the initial amount in the account is £5000 and the interest rate is 0.189% per month.

**(b)** Use the formula  for continuous compound interest to find the amount in the account after 8 months when the initial amount in the account is £5000 and the interest rate is 0.189% per month.

**(c)** When the bank calculates the interest rate in part **(a)** the amount is actually 1p more than your answer in part **(a)**. Without doing any calculations, explain how this could have happened. (Assume that the bank did not make a mistake!)

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| **(a)**  Amount is £5076.10**(b)**  Amount = £5076.17**(c)** At the end of each month the amount is rounded to the nearest penny. | This formula will be given in questions, when required.The answer must be corrected to two decimal places.This formula will be given in questions, when required. |

**Example 4 Drug concentration decay**

A patient has been on medication for some time. Just after taking the medication, the concentration of the drug in the patient’s blood is 0.45 mg/litre.

The concentration then decays and can be modelled by *A = C*e*–kt*, where *A* is the concentration in mg/litre and *t* is the number of hours since the medication was last given. After 6 hours the concentration is 0.32 mg/litre.

**(a)** State the value of constant *C*.

**(b)** Find the positive constant *k.*

**(c)** Write down the model for *A* in terms of *t*.

 For the medication to be effective, the concentration should not fall below 0.19 mg/litre.

**(d)** After how much time must the medication be given again?

It is found that the treatment is improved when the concentration of the medication is between 0.38 and 0.33 mg/litre.

**(e)** Find a model for the improved treatment.

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| **(a)** *C* = 0.45**(b)** 0.32 = 0.45 × e*–k*× 6  So **(c)** *A =* 0.45e*–*0.057*t***(d)** 0.19 = 0.45e*–*0.057*t* So  The medication must be given again after 15 hours.**(e)** *A =* 0.38e*–*0.057*t* | *C* is the initial amount. This is the concentration when *t* = 0. Substitute 0.32, 0.45 and 6.Solve the equation for *k*.Use the model.Substitute *A* = 0.19 and solve to find the time at which the concentration is 0.19 mg/litre.The decay stays the same, but the initial concentration is 0.38 |

**Note:** For the improved treatment, the amount of medication and the period before giving it again will be different, but you are not asked to find these.

Exercise

In the following questions, you should give answers to three significant figures when appropriate.

**1.** The mass, *m* grams, of a radioactive substance *t* years after the first observation is modelled by *m*= *a*e–0.02*t*, where *a* is a positive constant.

 The initial mass (when *t* = 0) is 300 grams.

 **(a)** State the value of *a* and express *m* in terms of *t*.

 **(b)** Find the amount of radioactive substance remaining 3 years after the first observation.

**2.** A scientist models the population, *P*, of a certain type of finch on a small isolated island by the equation , where *t* is the number of years after starting observations.

 **(a)** Explain why according to the model, the initial population is 600.

 **(b)** Explain why according to the model, the long-term population is 200.

 **(c)** Use the model to find the population after 1 year.

 The population decreased more slowly than the scientists expected and the population after 1 year is actually 500. The long-term population is not expected to change.

 **(d)** Refine the model, using the equation , where *k* is a constant to be found.

**3.** In this question use the formula  for continuous compound interest, where *P* is the amount invested in an account and *r*% is the interest rate. Find the amount in the account after 6 years when £10 000 was invested and the interest rate is 2.1% per year.

**4.** The graph shows the temperature of a cup of tea cooling.
The temperature *θ* is in °C and *t* is the time in minutes.
The temperature can be modelled by the equation where *c* and *k* are constants.

 **(a)** Find the values of *c* and *k*.

 **(b)** What is the temperature of the tea after 8 minutes?
 **(c)** What is the long-term temperature of the tea?

If the cup had been taken to a warmer room at the moment *t* = 0, the temperature could be modelled by the equation where *A*, *C* and *K* are constants.

 **(d)** Compare the constants *A* and *C* with the model for the cooler room.

**5.** Giles records the following measurements in an experiment.

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| --- | --- | --- |
| **Time, *t* minutes** | 0 | 5 |
| **Mass, *m* grams** | 100 | 80 |

 Giles believes his results can be modelled by , where *C* and *k* are positive constants.

 **(a)** Find the mass this model predicts for when *t* = 15.

 **(b)** Give a reason why there is not sufficient evidence to justify the use of this model.

Answers

**1. (a)** *a* = 300 and *m* = 300e–0.02*t*

 **(b)** *t* = 3 so *m* = 300e–0.02×3 = 282.529, so 283 grams of substance remains.

**2. (a)** The initial population is when *t* = 0,
 

 **(b)** As *t* tends to infinity, e–*t* tends to 0.

 

 **(c)** *t* = 1, 

 After 1 year there will be about 347 finches.

 **(d)** 

 So the model is .

**3.** 
 So the amount is £11 342.82

**4. (a)** From the graph: *θ* = 90 when *t* = 0
 90 = 21 + *c*e0,so *c* = 69
 From the graph: *θ* = 50 when *t* = 6
 

 **(b)** Use , .
 After 8 minutes the temperature is 43°C (or 42.8°C or 42.7°C)

 **(c)** 21°C

 **(d)** *A* > 21 since *A* is the long-term temperature and the room is warmer than 21°C.

 When *t* = 0, *θ* = 90 and *θ* = *A* + *C*e0 = *A* + *C*

 So *A* + *C* = 90 which means *C* < 69 since *A* > 21.

**5. (a)** *C* = 100, 80 = 100×e–*k*× 5, so *k* = 

 When *t* = 15, *m* = 100×e–0.0446*t* = 100×e–0.0446 × 15 = 51.2. So the mass is 51.2 g.

 **(b)** Two pairs of measurements are not enough to decide that the model is exponential.